PROPAGATION OF AN ELECTROMAGNETIC WAVE ACROSS A MAGNETIC FIELD IN A PARABOLIC PLASMA LAYER

V. R. Smilyanskii

The reflection and transmission coefficients are obtained, as well as the coefficient of transformation of an electromagnetic wave into a plasma wave. The problem of choosing the "physical" path of analytic continuation of the solutions is considered in the case of a wave equation with two poles.

1. Statement of the Problem. Let a plane wave be propagated along the z axis, while the plasma is likewise inhomogeneous along the z axis, and the external magnetic field is directed along the y axis. In this case, the electric field of the wave

$$E_x(z,\ t) = E_x(z)e^{i\omega t}$$

can be described by the equation [1]

$$\frac{d^2E_x}{dz^2} + \frac{\omega^2}{c^2}\left[1 - \frac{v(1-is-v)}{(1-is)^2 - u - (1-is)v}\right]E_x = 0, \quad i = \sqrt{-1} \tag{1.1}$$
$$v = g(1 - z^2/z_m^2), \quad g = \omega_k^2/\omega^2, \quad u = \omega_H^2/\omega^2, \quad s = \nu_{\text{eff}}/\omega$$

Here ω is the cyclic frequency; ω_H is the gyrofrequency for electrons; ω_k is the plasma frequency at the maximum of the layer; ν_{eff} is the effective number of collisions; z_m is the half-thickness of the layer; c is the velocity of light. Below we shall assume that u and s do not depend on z.

The propagation of the ordinary and extraordinary waves in a parabolic layer was considered in the geometric-optics approximation in [2]. However, in [2], the effect of regions where geometric optics is violated was not considered. The principal content of the subsequent analysis is precisely the consideration of the effect of poles in the coefficient of E_x on the propagation of the wave. The case of a linear layer was considered in [3, 4].

2. Asymptotic Solutions. Let us introduce the new independent variable $\tau = z/z_m$ in equations (1.1). Then this system of equations is written as

$$(\tau^2 - \tau_1^2)\frac{d^2E_x}{d\tau^2} + [p(\tau^2-1)^2 + q(2\tau^2 - \tau_1^2 - 1)]E_x = 0 \tag{2.1}$$
$$\tau_1^2 = 1 + \frac{u-(1-is)^2}{g(1-is)}, \quad q = \left(\frac{z_m\omega}{c}\right)^2, \quad p = \frac{qg}{1-is}$$

The two regular singular points of the equation ($\tau = \pm\tau_1$) merge into each other for $\tau_1 = 0$ ($s = 0$, $\omega^2 = \omega_H^2 + \omega_k^2$). For $\tau_1^2 = 1$ ($s = 0$, $\omega^2 = \omega_H^2$) the regular singular points are absent, which is quite understandable physically, since at the points $\tau = \pm 1$ the plasma density is equal to zero. As far as the value $\omega = \omega_H$ is concerned, this value of ω is not isolated in Eq. (1.1), which is associated with the approximation in which the equation was derived.

As is well known [5], solutions $E_x^{(1)}$, $E_x^{(2)}$ of Eq. (2.1) exist which have the following asymptotic representations for fulfillment of the conditions $|\tau| > |\tau_1|$ $|\sqrt{p\tau^2}| \gg 1$:

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 28-34, May-June, 1971. Original article submitted September 10, 1970.

$$E_x^{(1)} \sim (\sqrt{p}\tau^2)^{0.5r_1} e^{i0.5\sqrt{p}\tau^2} [1 + O(1/\sqrt{p}\tau^2)]$$

$$E_x^{(2)} \sim (\sqrt{p}\tau^2)^{0.5r_2} e^{-i0.5\sqrt{p}\tau^2} [1 + O(1/\sqrt{p}\tau^2)]$$

$$r_{1,2} = -\frac{1}{2} \mp \frac{p\tau_1^2 + 2q - 2p}{\sqrt{-4p}}$$

The condition $|\tau| > |\tau_1|$ leads to the condition $|\tau_1| < 1$, which restricts the range of frequencies considered. In particular, for s = 0, the allowable frequency range is

$$\omega_H^2 \leqslant \omega^2 < \omega_H^2 + 2\omega_k^2$$

If the incident wave propages from the direction $\tau < 0$, then $E_x^{(2)}$ for $\tau > 0$ describes the transmitted wave. Correspondingly, for $\tau < 0$ the quantity $E_x^{(2)}$ conversely represents the reflected wave and $E_x^{(1)}$ represents the incident wave.

In order to determine the amplitude coefficients of reflection (R) and transmission (D) it is necessary to know the relationship between the asymptotic solutions $E_x^{(2)}$ for $\tau > 0$, and $E_x^{(1)}$, $E_x^{(2)}$ for $\tau < 0$ ($|\tau| > |\tau_1|$). This relationship was established in [6] for a certain equation of which (2.1) is a particular case. Making use of the results of [6], one may write

$$D = {}^1/_2\pi^{-1}e^{i\pi(\eta-1)}\Gamma({}^1/_2 + \mu' - \eta)\,\Gamma({}^1/_2 - \mu' - \eta)$$
$$R = e^{i1.5\pi}(q_\pm 2\cos 2\pi\mu' + e^{-i2\pi\eta})\,D \qquad (2.2)$$
$$\eta = {}^1/_4(r_2 - r_1), \quad q_+ = 1, \quad q_- = 0$$

Here Γ is a gamma function; q_+ corresponds to bypassing of the singular points $\tau = \pm\tau_1$ (for transition from $\tau > 0$ to $\tau < 0$) along the upper half plane of the complex τ plane; q_- corresponds to bypassing of the singular points along the lower half plane; μ' is determined by the character of the singular points $\tau = \pm\tau_1$.

3. Determination of μ'. The solutions y_1, y_2 of Eq. (2.1) have the following form [5] in the neighborhood of the angular points $\tau = \pm\tau_1$:

$$y_1 = (\tau \mp \tau_1) \sum_{\nu=0}^{\infty} C_\nu(\pm\tau_1)(\tau \mp \tau_1)^\nu$$

$$y_2 = \mp b y_1 \ln(\tau \mp \tau_1) + \sum_{\nu=0}^{\infty} d_\nu(\pm\tau_1)(\tau \mp \tau_1)^\nu \qquad (3.1)$$

$$C_0 = 1, \quad d_0 = -1, \quad b = -(\tau_1^2 - 1)[p(\tau_1^2 - 1) + q]/2\tau_1$$

$$C_{\nu-1}(\tau_1) = -\frac{1}{2\nu(\nu-1)\tau_1} \sum_{k=k_0}^{\nu-2} C_k g_k(\nu), \quad k_0 = \begin{cases} 0 & (\nu \leqslant 6) \\ \nu - 6 & (\nu > 6) \end{cases} \quad (\nu = 2, 3, \ldots)$$

$$g_{\nu-2}(\nu) = (\nu-1)(\nu-2) + (\tau_1^2 - 1)[p(\tau_1^2 - 1) + q], \quad g_{\nu-5} = 4p\tau_1$$

$$g_{\nu-3}(\nu) = 4\tau_1[p(\tau_1^2 - 1) + q],$$

$$g_{\nu-4}(\nu) = 2[p(3\tau_1^2 - 1) + q], \quad g_{\nu-6}(\nu) = p$$

Here b is determined by the Frobenius method [5]. As is easily demonstrated from recurrent relationships, the coefficient C_ν is an even function of τ_1 if ν is even, and an odd function if ν is odd. As is evident from (3.1), the field E_x at the actual point of the pole ($\tau_1 \neq 0$) is finite. Making use of the procedure developed in [6], we have

$$\mu' = \frac{1}{8\pi i}\ln\delta + \frac{l}{4}, \quad \delta = \frac{2 - \beta + \sqrt{\beta(\beta - 4)}}{2 - \beta - \sqrt{\beta(\beta - 4)}} \qquad (3.2)$$

$$\beta = \left\{4\pi b\left[y_1 \frac{dy_1}{d\tau}\right]_{|\tau=0}\right\}^2$$

Here l is an integer which is to be determined. If the equation has only one regular singular point (or none at all), then $\mu' = {}^1\!/_4(\rho_1 - \rho_2)$ in accordance with [6], where $\rho_{1,2}$ is the solution of the defining equations for the case considered. For Eq. (2.1) we have

$$\mu'_{(\tau_1^2=0)} = {}^1/_4\sqrt{1 + 4z_m^2\omega_H^2 c^{-2}}, \quad \mu'_{(\tau_1^2=1)} = {}^1/_4 \qquad (3.3)$$

using the method indicated at the points $\tau_1^2 = 0$, $\tau^2 = 1$.

The integer l should be chosen in such a way that μ' from (3.2) coincides with μ' from (3.3) for $\tau_1^2 = 0, 1$. In order to determine l we choose $\tau_1^2 = 1$, since for such a value of the parameter τ_1^2 the solutions of Eq. (2.1) are single-valued and analytic throughout the entire domain $|z| < \infty$ (see Sec. 5). Let us confirm directly the fact that μ' from (3.2) for $\tau_1^2 = 1$ and μ' from (3.3) for $\tau_1^2 = 1$ coincide if $l = 1$. Thus, in (2.3), one should take $l = 1$.

If $s = 0$, then τ_1 is either real or purely imaginary ($\mathrm{Re}\,\tau_1 = 0$). Therefore, from the properties of symmetry relative to $C_\nu(\tau_1)$ indicated above, and likewise from (3.1), (3.2), it follows that for $s = 0$, $\beta \geq 0$.

Let $s = 0$ and $0 \leq \beta \leq 4$. Then [see (3.2)]

$$|\delta| = 1, \quad 0 \leqslant \arg\, \delta \leqslant 2\pi, \quad {}^1/_4 \leqslant \mu' \leqslant {}^1/_2$$

i.e., for the indicated variation of β the variable under the logarithm sign bypasses the branching point of the logarithm and the given branch goes over into another. Therefore, for $\beta > 4$ one should take $\ln\delta + i2\pi$.

Let $s = 0$, $\beta \geq 4$. In this case,

$$\arg\, \delta = 0, \quad \mu' = {}^1/_2 + i\,\psi \quad (\psi > 0)$$

Making use of the recurrent relationship for C_ν, it may be shown that regardless of what side $\tau_1 \to 0$ takes place from,

$$\lim_{\tau_1 \to 0} \frac{y_1(\tau = 0)}{\tau_1} = \sum_{k=0}^{\infty} a_k, \quad \lim_{\tau_1 \to 0} \frac{dy_1(\tau = 0)}{d\tau} = \sum_{k=0}^{\infty} (k+1)\, a_k$$

$$a_k = (-1)^k\, [2^k k!\, (k+1)!]^{-1} \prod_{\nu=2}^{k+1} \left[(\nu - 1)(\nu - 2) - \left(\frac{z_m \omega_H}{c}\right)^2\right]$$

and, consequently, for μ' from (3.2) there exists

$$\lim_{\tau_1 \to -0} \mu' = \lim_{\tau_1 \to +0} \mu'$$

At the same time,

$${}^1/_4 \leqslant \mathrm{Re}\,\mu' \leqslant {}^1/_2 \quad \text{for} \quad s = 0 \qquad (l = 1)$$

On the other hand, for a fairly thick layer the expression from (3.3) yields

$$\mu'_{(\tau_1^2 = 0)} \approx 0.5 z_m \omega_H / c \gg 1$$

From this it follows that the function $\mu'(\tau_1^2)$ has a discontinuity at the point $\tau_1^2 = 0$ ($s = 0$, $\omega^2 = \omega_H^2 + \omega_k^2$), which can be eliminated. This can evidently be explained by the fact that the original equation (1.1) was derived without consideration of space dispersion. Of course, there is no basis for expecting an actual step discontinuity $R(\mu')$, $D(\mu')$ at the point $\tau_1^2 = 0$.

4. Obtaining R, D, and the Transformation Coefficient. Let $s = 0$. In this case,

$$\eta = i\varkappa \qquad (\varkappa = 0.25 z_m\, (c\omega_k)^{-1}\, (\omega_k^2 - \omega_H^2 - \omega^2))$$

Assume additionally that $0 \leq \beta \leq 4$. Then μ' is real. For this case we have from (2.2) the following results in accordance with Eq. (8.344.2) in [7]:

$$|D|^2 = \frac{e^{-2\pi\varkappa}}{2\cos 2\pi\mu' + e^{2\pi\varkappa} + e^{-2\pi\varkappa}} \tag{4.1}$$

$$|R|^2 = (q_{\pm}2\cos 2\pi\mu' + e^{2\pi\varkappa})^2 |D|^2$$

A simple check of the derived formulas shows that for $1/4 \leq \mu' \leq 1/2$ we have $|R|^2 + |D|^2 \geq 1$ for bypass along the lower half plane, and $|R|^2 + |D|^2 \leq 1$ for bypass along the upper plane, $|R|^2 + |D|^2 = 1$ holding in both cases solely for $\mu' = 1/4$. From this it follows that the physically correct result yields a bypass along the upper half plane.* Therefore, the correct expression for R and D in the general case is

$$D = {}^1/_2\pi^{-1}e^{i\pi(\eta-1)}\Gamma({}^1/_2 + \mu' - \eta)\,\Gamma({}^1/_2 - \mu' - \eta)$$
$$R = e^{i1.5\pi}(2\cos 2\pi\mu' + e^{-i2\pi\eta})\,D \tag{4.2}$$

Let us likewise consider the case s = 0, $\beta \geq 4$. In this case we have

$$|D|^2 = \frac{1}{2}\frac{(\psi - \varkappa)}{(\psi + \varkappa)}\frac{e^{-2\pi\varkappa}}{[\mathrm{ch}\,2\pi\psi - \mathrm{ch}\,2\pi\varkappa]}$$
$$|R|^2 = (e^{2\pi\varkappa} - 2\,\mathrm{ch}\,2\pi\psi)^2\,|D|^2$$

on the basis of Eqs. (8.331) and (8.322.1) in [7].

If s = 0, $\eta = 0$, then Eq. (4.2) can be simplified substantially; specifically,

$$D = -\frac{1}{2\cos\pi\mu'}, \qquad R = -i\,(2\cos 2\pi\mu' + 1)\,D$$

If in (4.1) we formally place $2\pi\mu' = 0$ ($\mu' = 1/4$), then the dependence of the derived formal expressions for $|R|^2$, $|D|^2$ on the frequency ω is exactly the same as it is in the case of normal incidence of an electromagnetic wave on a parabolic layer of isotropic plasma (see [1], Sec. 17). Therefore, the general character of this dependence for $|R|^2$ and $|D|^2$ is not altered substantially even for small $|\cos 2\pi\mu'|$ (i.e., for small $\mu' - 1/4 > 0$).

However, the substantial difference from the case of normal incidence of a parabolic layer of isotropic plasma is manifested here in the fact that the coefficient of transformation of an electromagnetic wave into a plasma wave $|F|^2 = 1 - |R|^2 - |D|^2$ (see below) is not equal to zero.

As is evident from (3.2), the quantity β (and thus μ' also) will, in general, be a complex function of ω^2, ω_k^2, ω_H^2, z_m^2, c^2. However, under specified conditions the expression for β may be simplified substantially. Let us make use of the expansion

$$(y_1 dy_1 / d\tau)^2 = (\tau \mp \tau_1)^2[1 + 6C_1(\pm\tau_1)(\tau \mp \tau_1) + \ldots] \tag{4.3}$$

Let s = 0, $\tau = 0$. Then the second term of the series in (4.3) may be neglected in comparison with the first term, if

$$\omega^2 \ll \omega_H^2 + \omega_k^2\,\frac{2}{3}\left(\frac{c}{\omega_H z_m}\right)^2$$

In this case,

$$\beta \approx 4\pi^2\left(\frac{\omega^2 - \omega_H^2}{\omega_k^2}\right)^2\left(\frac{z_m\omega_H}{c}\right)^4 \tag{4.4}$$

If, for example, $\omega_H^2 \ll \omega_k^2$ and $2/3(c/\omega_H z_m)^2 \gg 10$, then Eq. (4.4) may be used throughout the entire interval of allowed frequencies $\omega_H^2 \leq \omega^2 < \omega_H^2 + 2\omega_k^2$ (see Sec. 2).

From (4.2) it follows that the phase shift φ between the reflected and infinite waves at the beginning of the layer ($\tau = -1$) is

$$\varphi = -\operatorname{Im}(\eta\ln p) + \operatorname{Re}\sqrt{p} - \arg R \tag{4.5}$$

From (4.5) we have the following result in accordance with Eq. (8.362.1) from [7]:

$$\Delta t_{bo} = \frac{d\varphi}{d\omega} = -\frac{d}{d\omega}[\operatorname{Im}(\eta\ln p) + \arg(2\cos 2\pi\mu' + e^{-i2\pi\eta}) + \pi\operatorname{Re}\eta] -$$
$$-2\operatorname{Im}\left\{\mu'\frac{d\mu'}{d\omega}\sum_{k=0}^{\infty}\frac{1}{\chi_k} + \left[0.577215 + \frac{\sigma}{\chi_0} - \sum_{k=1}^{\infty}\frac{\chi_0 + k\sigma}{k\chi_k}\right]\frac{d\eta}{d\omega}\right\}$$
$$\sigma = {}^1/_2 - \eta, \qquad \chi_k = (\sigma + k)^2 - \mu'^2$$

*It is of interest that in [6] the physically correct result was obtained for bypass along the lower half plane.

The quantity $|F|^2 = 1 - |R|^2|D|^2$ for $s = 0$ characterizes the relative fraction of energy which is absorbed in the region of the pole. In the region of the pole ($1 - v - u = 0$, $s = 0$) transformation of the electromagnetic wave into a plasma wave occurs (see [1, 8]) and, therefore, for waves having a small amplitude the loss of energy in the region of the pole for $s = 0$ may be explained by transformation.

On the other hand, the pole itself develops if space dispersion is neglected and, consequently, in the limiting case it reflects the influence of the discarded terms. Therefore, for $s = 0$ the quantity $|F|^2$ may be treated as the coefficient of transformation of an electromagnetic wave into a plasma wave at small values of the parameter β_T^2 which characterizes the space dispersion. In other words,

$$|F|^2_{s=0} = \lim L \quad \text{for} \quad \beta_T^2 \to 0$$

where L is the transformation coefficient [i.e., $|F|^2_{s=0}$ is the first term of the expansion of $L(\beta_T^2)$ into a series in β_T^2]. Unlike [8] (where a different law for the variation of the electron density is also taken), the transformation coefficient here can be determined from a second-order equation.

The treatment of the quantity $|F|^2$ presented above has been considered in various aspects in [9, 11].

If $s = 0$, $0 \leq \beta \leq 4$, then

$$|F|^2 = \frac{|\cos 2\pi\mu'| - 2e^{-2\pi\varkappa}\cos^2 2\pi\mu'}{\operatorname{ch} 2\pi\varkappa - |\cos 2\pi\mu'|} \tag{4.6}$$

(Here $\cos 2\pi\mu' < 0$.). The dependence of $|F|^2$ on ω^2/ω_k^2, which is given in (4.6) for $(c/z_m\omega_H^2) = 20$, $\omega_k^2 = 5\omega_H^2$, is displayed in Fig. 1.

In the given paper the choice of the path of analytic continuation of the solutions is achieved from the condition governing the energy dissipation ($|R|^2 + |D|^2 \leq 1$). Such an approach in the case of one simple pole has already been used previously (for example, in [9, 10]). However, the case of two poles has certain specific peculiarities. Therefore, it is expedient to discuss it in greater detail (see Sec. 5).

5. On the Choice of the "Physical" Path of Analytic Continuation of the Solution. Equation (2.1) is a particular case of the equation

$$\frac{d^2y}{dz^2} + \left(\frac{Q_1 z}{z^2 - z_1^2} + a_1 z\right)\frac{dy}{dz} + \left[\frac{Q_2 z^2}{(z^2 - z_1^2)^2} + \frac{Q_3}{z^2 - z_1^2} + a + bz^2\right] y = 0 \tag{5.1}$$

considered in [6], where Q_1, Q_2, Q_3, a, a_1, and b are arbitrary constants.

The solutions of Eq. (5.1) are considered in the present paper and in [6] basically in the domain $|z_1| < |z| < |z_m| < \infty$. In principle, more than two paths for performing the analytic continuation of these solutions from the positive half axis onto the negative one are possible in the z plane (for example, between the poles $z = \pm z_1$ also). Therefore, let us consider the question of which paths, in general, of analytic continuation may lead to a physically correct result. We shall conduct the consideration in parallel for the present paper and for [6]. Hereafter we shall add the letter A to the numbers of the equations from [6] [for example, (A.1.2)].

As is well known [1], consideration of the space dispersion leads to fourth-order systems instead of equations (1.1), (A.1.2); the coefficients of these systems (and, therefore, the solutions) are analytic and single-valued in the neighborhood of the zeros of $\delta = [(1 - is)^2 - u - (1 - is)v]$, ε'. The equations (1.1), (A.1.2) are derived from equivalent systems (in the domain considered) of fourth-order equations if the small parameter β_T^2 in the senior derivatives is placed equal to zero (i.e., if the equations are degenerate in a definite sense).

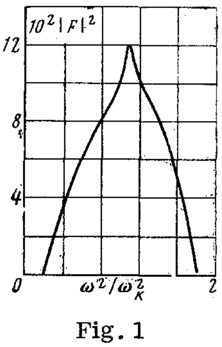

Fig. 1

The effect of the discarded terms is especially great in the region where the coefficient of the second derivative is small (i.e., near the zeros of δ, ε'). Therefore, in general, the solutions of the degenerate equations on the real axis in the neighborhood of the zeros of δ, ε' (if the zeros are situated on or near the real axis) may not correspond to the actual physical picture, i.e., they may not be physical. This is indicated by the ambiguity of the solutions and their divergence at the zeros. In this case, the physical solutions in the neighborhood of the zeros of δ, ε' may be obtained only by means of a nondegenerate fourth-order system.

A reliable exception is the case when the solutions of the degenerate equation are unique and analytic in the neighborhood of the zeros of δ, ε', notwithstanding the presence of the pole. [Note that the solutions of (A.2.1) ($z_1 = 0$) are unique and analytic, that is, they are physical, throughout the entire domain $|z| < \infty$, including at the zero of $\varepsilon'(z = 0)$. This derives from (A.2.2).] In particular, $G^{(1,2)} = \text{const}$ for $z \to 0$. The influence of the discarded term is slight here even in the neighborhood of the zero of ε'. This is likewise indicated by the absence of transformations ($|R|^2 + |D|^2 = 1$, see Sec. 2).

Consequently, the correct way of performing analytic continuation of the solutions of Eqs. (1.1), (A.1.2) (taken far from the zeros of δ, ε') from the positive z half axis to the negative half axis in this case must not necessarily coincide with the real axis (or be congruent with it via continuous deformation) in the neighborhood of the zeros of δ, ε'. Moreover, it is natural to expect that, in the case considered, the bypassing of the zeros along a path situated sufficiently far from them in the complex plane (i.e., where the influence of the discarded terms is small) is precisely what yields a physically correct result. (It would be desirable to indicate a similar device in the quasiclassical method [12].)

It is precisely such waves that were used in the given work and in [6]. In these cases the path between the poles $z = \pm z_1$ (i.e., between the zeros of δ, ε') will not be simple, since for specific values of the parameters in (2.1) and (A.3.1) the two poles (the zeros of δ, ε') are close and even merge into one.

Note likewise that in order to determine the integer number l included in μ', parameters were chosen in Eqs. (2.1) and (A.3.1) such that the solutions of these equations were unique and analytic in the entire domain $|z| < \infty$. After all that has been said above, this does not require clarification.

LITERATURE CITED

1. V. L. Ginzburg, Propagation of Electromagnetic Waves in a Plasma [in Russian], Nauka, Moscow (1967).
2. I. C. N. Scott, "The Poynting vector in the ionosphere," Proc. IRE, 38, No. 9 (1950).
3. N. G. Denisov, "The effect of a constant magnetic field on the resonance effect observed as a result of the reflection of an electromagnetic wave from an inhomogeneous plasma," Radiotekhnika i Élektronika, 1, No. 6 (1966).
4. N. G. Denisov, "On the absorption of radio waves in resonant domains of an inhomogeneous plasma," Radiotekhnika i Élektronika, 4, No. 3 (1969).
5. E. L. Haynes, Ordinary Differential Equations [in Russian], Gostekhizdat Ukrainy, Khar'kov (1939).
6. V. R. Smilyanskii, "Oblique incidence of an electromagnetic wave on a parabolic plasma layer," Zh. Prikl. Mekh. i Tekh. Fiz., No. 3 (1969).
7. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, Moscow (1963).
8. T. H. Styx, "Radiation and absorption via mode conversion in an inhomogeneous collision-free plasma," Phys. Rev. Letters, 15, No. 23 (1965).
9. A. D. Piliya, "On the transformation of waves in a weakly inhomogeneous plasma," Zh. Tekh. Fiz., 36, No. 11 (1966).
10. A. D. Piliya and V. I. Fedorov, "Linear transformation of waves in an inhomogeneous magnetically active plasma," Zh. Éksperim. i Teor. Fiz., 57, No. 4 (1969).
11. N. S. Erokhin, "On the problem of the anomalous transformation of waves in an inhomogeneous plasma," Ukrainsk. Fiz. Zh., 14, No. 12 (1969).
12. L. D. Landau and E. M. Lifshits, Theoretical Physics, Vol. 3. Quantum Mechanics. Nonrelativistic Theory [in Russian], Fizmatgiz, Moscow (1963).